A MATHEMATICAL MODEL FOR GROUP STRUCTURES

by

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INTRODUCTION

Many kinds of theories have been developed to explain human behavior. They may be classified in many ways - in terms of outstanding men, of schools of psychology, in terms of the emphasis placed upon certain concepts or areas of study. A way of classifying which is useful for the purpose of this paper is one which differentiates between a) theories which explain human behavior as a function of factors which may be coexistent but independent of each other, and b) theories which explain human behavior as a function of groups of factors constituting a continuously interacting field.

At the time of the first world war psychologists in Germany were splitting roughly into these two camps: One group followed the path of breaking down the person and the situation into elements and attempting to explain behavior in terms of simple causal relationships. The other group attempted to explain behavior as a function of groups of factors constituting a dynamic whole - the psychological field. This field consisted essentially of the person himself and his environment as he saw it. In these terms, the problem was no longer conceived as one of relationships between isolated elements, but one of dynamic interplay of all the factors of the situation.

At this time Kurt Lewin began to formulate a method of analysis of psychological situations which rested upon their restatement in mathematical terms: geometry for the expression of the positional relationships between parts of the life space, and vectors for the expression of strength, direction, and point of application of psychological forces. The use of geometry was natural in a psychological approach which insisted upon a world "as the person himself sees it", since human beings tend to picture the contextual field as existing in a "space" around them. Also, the geometric approach offered a convenient means for diagramatic representation of many psychological situations.[1] The most important and, in a sense, the only reason for the use of geometry lay in the fact that the assumption of groups of interrelated factors implied the existence of a mathematical space and some means of handling it was necessary. The task of representing the relationships between groups of psychological data laid certain requirements upon the type of geometry that could be used. It had, in the first place, to be a geometry that did not rest upon a groundwork of assumptions impossible to satisfy. Also, in order to be immediately useful, it had to be a geometry that pre-supposed no greater possibilities of measurement and definition than were present in psychology at the time.

A sufficiently generalized and non-metric geometry was found in topology. Certain fundamental notions from topology - "connectedness", "region", "boundary", - showed promise in handling spatial relationships in psychology. These ideas were borrowed and put to use although, for the most part, the assumptions underlying them could not be related to psychology. That these concepts from topology proved useful is beyond question if one may judge from the experimental settings which they stimulated and made possible to formulate. Lewin and his students pioneered in fields previously considered too difficult to approach experimentally.[2]

1. Actually, Lewin never intended to use geometry as a means for producing diagrams or illustrations. His *geometrical representations* were not analogies to be dismissed whenever their implications became inconvenient. They were intended to be mathematical statements of relationships: not statements of "what the situation is like", but statements of "what the situation is".

2. Barker, R., Dembo, T., and Lewin, K., Frustration and regression: An experiment with young children, *Univ. Ia. Stud. Child Welf.*, 1941, 18.
Dembo, T., Der Arger als dynamisches Problem, *Psychol. Forsch.*, 1931, 15, 1-44.
Kounin, Jacob S. Experimental Studies in Rigidity, *Character and Personality*, 1941, 9, 273-282.
Lewin, K., A Dynamic Theory of Personality, New York, McGraw-Hill, 1935.
-----, Principles of Topological Psychology, New York McGraw-Hill, 1936.
Lippitt, R., An *analysis of group* reaction to three types of experimentally created social climate, Iowa City, Univ. of Iowa, Ph.D. Thesis, 1940.
Mackinnon, Donald W., A Topological Analysis of Anxiety, *Character and Personality*, 1944, 12, 163-176.
Mahler, V., Ersatzhandlungen verschiedenen Reallitatsgrades, *Psychol. Forsch.*, 1933, 18, 26-89.

Concepts borrowed out of their context, however, useful though they may be in specific instances of application, rarely yield full value until they are related to each other in some systematic way. The coordination of the formal body of topological geometry with psychological concepts seemed hopeless. What seemed possible was the definition of a geometry which retained the demonstrated advantages of the concepts borrowed from topology, and which seemed suitable as a system to the uses of psychological analysis. Lewin attempted this task by introducing a geometry which he called "hodology." In *hodology*, primarily designed to treat problems of distance and direction, a basic assumption is that between *any two parts* of a whole there can be distinguished a "shortest path."

> "In Euclidian space, both *distance and direction* refer to a connection between two points. From the point of view of topology, this connection is to be considered a 'path' from a to b. There are of course many paths possible between a and b. *One of these possible* paths is distinguished as the shortest path between a and b...... Since topology does not know *differences of size*, one cannot use shortness as a general principle for such a selection. It might be possible to substitute for the concept 'shortest path' the idea of the minimum number of regions which might be crossed by a path from a to b. However, we shall not confine ourselves to this one method of selecting a *distinguished path from* the possible ones."[3]

From this idea of "shortest path" Lewin develops an analytical approach to the properties of wholes. In this paper, he points to the importance of the specific pattern that cells in a whole may take, and he introduces certain concepts regarding *regions of special properties* in such patterns. Assuming "natural wholes where all boundaries have the same strength" he *distinguishes regions in the* structure from which a change would spread most quickly to all other parts, and regions from which such spreading would take longest to cover the entire structure. Also, he distinguishes areas in the structure most and least susceptible to changes "on the outside." Lewin points out, also, that these concepts are relevant to both *problems* of cognitive structure - the organization of mental materials - and problems of group structure - the organization of individuals in social groups.

To go back to the definition of "shortest path", upon which all of the preceding discussion is based, we find that Lewin indicates several possible definitions:

> "The property which makes one path between two regions of the life space outstanding (the 'shortest path') seems to vary considerably with the situation. Sometimes the fastest connection is outstanding; at other times it is the cheapest connection, or the most pleasant, or the least dangerous. So, all that we would like to assume is that there *should be, in the given case, one* outstanding path from a to b."[4]

The criteria "most pleasant" or "least dangerous" introduce dynamic factors in the problem of defining a shortest path. The criterion "minimal regions to be crossed by a path from a to b", however, is a purely positional one. It seems that only upon the latter type of criterion can one develop mathematical statements of distance and direction as functions of the number and position of the parts in the whole. In this paper, the "minimal regions" criterion for the distinguishing of shortest paths is accepted and an attempt is made to explore in a limited way the consequences of such an assumption. Also, an *attempt is made to make changes in or additions* to accepted definitions which might permit a more flexible use of concepts developed therefrom. Such changes are not "arbitrary" but are made with an eye to problems in psychology which might thus be handled more easily. In some cases changes which would make certain problems more amenable to mathematization have been avoided because of the complications they introduce in the initial formulation of concepts. A case *in point is that of the transitiveness* of the "touching" relationship between regions or cells. One of the assumptions made below is that if cell A_1 is said to be touching cell A_2, then cell A_2 may be said to be touching cell A_1." This assumption was *made* in order to secure certain advantages which will be apparent in the material that follows. But other *advantages were foregone. Under the condition* of the assumption as stated, a chain of cells may be regarded as a path along which a change of state might spread stepwise from cell to cell - from any cell to any neighboring cell. If one thinks of the chain of cells as representing a sequence of stages through which a person might pass, then one must conclude *that the sequence of stages is reversible at any point* - a fact not always or often true in human experience.

3. Lewin, Kurt, The Conceptual Representation and the Measurement of Psychological Forces", *Contributions to Psychological Theory*, I, Duke University Press, 1938, p. 25.

4. *Ibid*, p. 25.

If the assumption of transitivity were not made, or explicitly rejected, then it would be mathematically possible to describe chains of cells or paths with very different properties of connectedness. Our hypothetical person might in such a case pass from cell A to B to C but not be able to retrace his steps. The path might be irreversible entirely or in part.

These concepts of pattern and communication are the heart of this paper. They are developed with the deliberate purpose of application to psychological situations. Although no rigid coordination of these ideas with psychological or social situations is attempted, general areas within which application might be fruitful are suggested. For instance, in the realm of social groups, there would seem to be two outstanding aspects of communication deserving attention: that of communication between individuals (or between groups), and that of communication between ideas and attitudes. The spread of rumor is a good example of the close relationship between these aspects. While the rapidity, direction, and extent of the spread depends partly upon the patterns of connection between individuals and groups, they also depend - especially with respect to the growth and bias of the rumor, and the readiness to hear and transmit - upon the connection of the content of the rumor with other ideas and attitudes.

BASIC ASSUMPTION

1. The space being dealt with consists of collections of cells.
2. A cell is equivalent to a point or position in the space.
3. A given cell may or may not be touching another cell.
4. If a cell A_1 is touching another cell A_2, then cell A_2 is said to be touching cell A_1.
5. A cell cannot touch itself.

DEFINITIONS

1. Boundary of a cell: the boundary of a cell A consists of all cells touching A.
2. Region: a region is any class or collection of cells
3. Open cell: cell A is open relative to region g if the boundary of A is not contained in g.
4. Closed cell: cell A is closed relative to region g if the boundary of A is contained in g.
5. Boundary of a region: the boundary of the region g is the class of all cells not in g and touching at least one cell in g.
6. Chain: cells $A_1, A_2, \ldots\ldots, A_n$ are said to form a chain if A_1 is touching A_2, A_2 is touching A_3, ..., A_{n-1} is touching A_n. A_n may or may not be equivalent to A_1.
   a) Simple chain: cells $A_1, A_2, \ldots\ldots, A_n$ are said to form a simple chain if A_1 touches cell A_2 and no other cell, if A_2 touches A_1 and A_3 and no other cell, if A_3 touches A_2 and A_4 and no other cell, etc., ...and cell A_n touches cell A_{n-1} and either does or does not touch cell A_1.
7. Length of a chain: the length of a chain is equal to the number of cells contained in the chain less one.
8. Structure: a region g is said to be a structure if for any pair of cell A_1, A_2 contained in g, there exists a chain contained in g and connecting A_1 and A_2.
9. Distance between two cells: the distance between any two cells A_1, $A_2(\overline{A_1A_2})$ in a structure w is the minimum length chain contained in w and connecting A_1 and A_2.
10. Distance between a cell and a region: when cell A and region g are both included in w, and when A is not in g two distances are distinguished.
    a) Maximum distance: the longest distance of all distances from A to every cell in g.
    b) Minimum distance: the shortest of all distances from cell A to any cell in g.
11. The outermost region of a structure: the outermost region of a structure is the class of all cells which are open relative to the structure.
12. The innermost region of a structure: the innermost region of a structure is the class of all cells with the largest minimum distance from the outermost region of the structure.
13. The largest of the maximum distances between the outermost and innermost regions of a structure is denoted by the letter r.
14. The largest of all distances between a cell A_1 and any other cell in the structure is denoted by p.
15. The diameter of a structure: the diameter of a structure (d) is equal to the largest p that can be found in the structure.
16. The central region of a structure: the central region of a structure is the class of all cells with the smallest p to be found in the structure.
17. The peripheral region of a structure: the peripheral region of a structure is the class of all cells having the greatest maximum distance from the central region. This distance will be denoted as c.

A DISCUSSION OF THE DISTANCES d, c, and r.

A Method of Illustration.

In a discussion of patterns of communication within a structure, three structural distances are distinguished: d, c, and r (see Definitions 12,14, & 17). Although not strictly necessary, and often undesirable, some kind of picture of a structure is helpful in illustrating certain relationships between these distances. The pictures will be constructed in the following way: a structure of three cells all of which touch each other would be shown as in Figure 1.

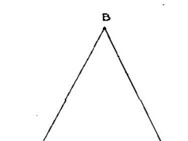


Figure 1

If in this structure cell A was an open cell (see Definition 3), the structure would be shown as in Figure 2.

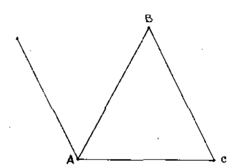


Figure 2

The picture shown in Figure 3 would mean

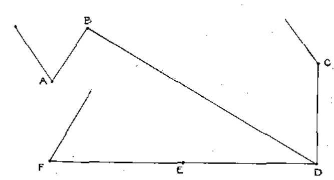


Figure 3

1) That A, C, F were open cells and that B, D, E were closed cells; 2) That A (with respect to this structure) touched only B, and F touched only E; that B touched A and D, D touched B, C, and E, and E touched D and F.

The reason for this change from the kind of pictures used by Lewin is that certain types of structures are very difficult or impossible to draw in that manner. For instance, it is impossible to represent, Lewin-wise, a structure in which A touches B, B touches C, and A and C are open cells and B is closed. (see Figure 4)

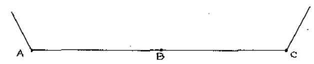
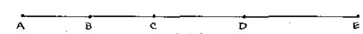


Figure 4

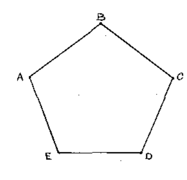
The distance d (Definition 14: The largest of all distances between a cell A_1 and any other cell in the structure is denoted by p; Definition 15: The diameter of a structure (d) is equal to the largest p that can be found in the structure.)

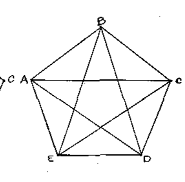
The way in which the distance d may vary in a structure with a constant number of cells can be shown by the use of several pictures.



d = 4

Figure 5a





d = 2

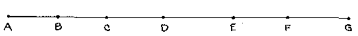
Figure 5b

d = 1

Figure 5c

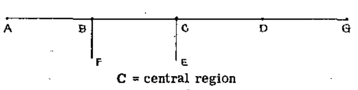
It may be seen from the foregoing illustrations that the limits of the distance d may be expressed in terms of n (the number of cells in the structure). The value of d will be at its minimum when the longest distance which may be found between any two cells is equal to n-(n-1) or simply 1, as in Figure 5c. The value of d will be at a maximum when the longest distance to be found between any two cells is equal to n-1 - which means that the structure will take the form of a simple chain in which the first cell does not touch the last one. The distance c (Definition 17:) (The peripheral region of a structure is the class of all cells having the greatest maximum distance from the central region. This distance will be denoted as c.)

The way in which the distance c may vary in a structure with a constant number of cells is shown in the following pictures.



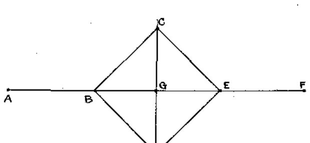
D = central region
$c = 3$

Figure 6a



C = central region
$c = 2$

Figure 6b



C + D + G = central region
$c = 2$

Figure 6c

In Figure 6c the peripheral region (see Definition 16) would consist of cells A and F. Lewin defines the peripheral region of a structure as all cells A for which a cell B may be found so that the shortest distance from A to B is equal to d. In some cases, such as that shown in Figure 6c, both definitions distinguish the same region. In other cases, such as that shown in Figure 6d, the regions distinquished are different.

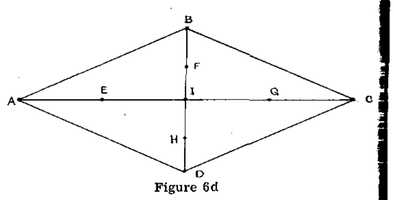
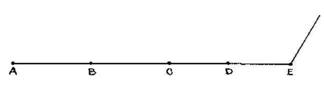


Figure 6d

According to Lewin's definition the peripheral region consists of cells A,B,C,D,E,F,G,H. According to the definition in this paper the peripheral region consists of cells A,B,C,D.

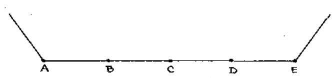
The distance r (Definition 13: The largest of the maximum distances between the outermost and innermost regions of a structure is denoted by the letter r.)

The way in which the distance r may vary in a structure with a constant number of cells is shown in the following pictures.



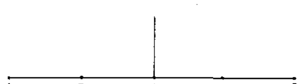
A = innermost region
$r = 4$

Figure 7a



C = innermost region
$r = 2$

Figure 7b



A + E = innermost region
$r = 2$

Figure 7c

In addition to showing how the distance c may vary, Figures 7a, 7b, 7c illustrate some of the properties of a region as they have been defined (see Definition 2). In Figure 7c, for instance, the region distinguished as innermost consists of two non-connected cells as far apart as it is possible for them to be.

The Limits of the Values That c and r may Assume.

The limits of the values that c and r may assume can be expressed in terms of d in the following way.

$d = c = d/2$[5] $\qquad d = r = d/2$

In other words, the limits of the values that c and r may assume are d at one extreme and 1/2 d at the other.

In order to give a proof of these two statements, it is necessary to state a general principle with respect to distances within structures. The principle is that given three cells, A, B, and C in a structure, $\overline{AB}$ (the distance A to B, see Definitions 6, 6a, 8) will be equal to or less than $\overline{BC}$ plus $\overline{CA}$. Proof of this general principle may be developed from the assumptions and definitions:

1) If cell C is contained in the chain $\overline{AB}$, then $\overline{AC} + \overline{CB} = \overline{AB}$.

2) If the cell C is not contained in the chain $\overline{AB}$, then the chain $\overline{ACB}$ is by definition equal to or longer than the chain $\overline{AB}$.

3) Therefore, $\overline{AB} \leqq \overline{AC} + \overline{CB}$

Returning to the question of the limits of c and r in terms of d.

1) Let A be a cell in the central region.
2) Let $\overline{CB} = d$.
3) $\overline{AB} + \overline{AC} \leqq 2c$.
4) $\overline{AB} + \overline{AC} \geqq d$.
5) Therefore, $2c \geqq d$
or $d \geqq c \geqq d/2$.

And in the same way with respect to r.

1) Let A be a cell in the innermost region.
2) Let $\overline{CB} = d$.
3) $\overline{CA} + \overline{AB} \leqq d$.
4) $\overline{CA} + \overline{AB} \geqq d$.
5) Therefore, $2r \geqq d$.
or $d \geqq r \geqq d/2$.

Although c and r have the same limits, they do not vary in the same way. Although formal proof of this has not been worked out, structures may be drawn in which c and r do not have the same values. (Figure 8).

$d = 4$ $\qquad c = 2$ $\qquad r = 3$

Figure 8

A glance at Figures 5a,b,c, and 6a,b,c, is sufficient to show that in a structure of n cells the way that d and c will vary within their limits depends upon the number and pattern of interconnections between the cells. This problem was approached in two ways, and although the objective of finding the function along which d and c vary was not attained, the results are interesting enough to be stated.

The Limits of si_x[6] for a Structure of n Cells.

1) In a structure in which every cell touches every other cell $i_x = n-1$ and $si_x = n(n-1)$.

2) In a structure in which each cell has the fewest possible cells touching it, the structure will take the form of a simple chain in which cell A_1 does not touch cell A_n.

3) Therefore $i_{A_1} = 1, i_{A_2} = 2, i_{A_3} = 2, \ldots\ldots,$ $i_{A_{n-1}} = 2, i_{A_n} = 1$.

4) Therefore $si_x = 2(n-1)$.

5) Therefore, we may say that for a structure of n cells the value of si_x will vary from n(n-1) to 2(n-1), the range of variation being $n(n-1)-2(n-1) = (n-1)(n-2)$.

The Limits of $\sum se_{xy}$[7] for a Structure of n Cells.

1) In a structure in which every cell touches every other cell, $se_{A_1x} = n-1$ and $\sum se_{xy} = n(n-1)$.

2) In a structure in which each cell touches as few other cells as possible, the structure will take the form of a simple chain in which cell A_1 does not touch A_n.

3) Therefore,
$se_{A_1x} = 1 + 2 + 3 + \ldots\ldots + n-1$, and

5. In reading the value of d/2, read fractions as the next higher integer. The quantity d/2 refers to a distance in a structure and distances are so defined that a fractional distance has no meaning.

6. i_A = the number of cells touching cell A.
si_x = the sum of i for every cell in the structure--that is, $i_{A_1} + i_{A_2} + i_{A_3} + \ldots\ldots + i_{A_n}$.

7. $e_{A_1A_2} = \overline{A_1A_2}$; se_{A_1x} = the sum of all distances between cell A_1 and every other cell in the structure; $\sum se_{xy}$ = the sums of all distances between all pairs of cells in the structure.

$se_{A_2x} = 1 + 1 + 2 + \ldots\ldots + n-2$, etc., to

.
.
.
.

$se_{A_1x} = n-1 + \ldots\ldots\ldots + 2 + 1.$

4) $se_{A_1x} = \frac{n(n-1)}{2}$

$se_{A_2x} = \frac{(n-1)\ (n-2)}{2} + \frac{n-(n-2)\ n-(n-1)}{2}$

$se_{A_3x} = \frac{(n-2)\ (n-3)}{2} + \frac{n-(n-3)\ n-(n-2)}{2}$

.
.
.
.

$se_{A_{n-2}x} = \frac{n-(n-3)\ n-(n-2)}{2} + \frac{(n-2)(n-3)}{2}$

$se_{A_{n-1}x} = \frac{n-(n-2)\ n-(n-1)}{2} + \frac{(n-1)(n-2)}{2}$

$se_{A_nx} = \frac{n(n-1)}{2}$

5) Therefore $\sum se_{xy}$ may be expressed as $n(n-1) + (n-1)(n-2) + (n-2)(n-3) + \ldots\ldots + n-(n-1)\ (n-n)$ or simply

$$\frac{n(n^2-1)}{3}$$ [8]

6) Therefore, the range of values for $\sum se_{xy}$ in a structure of n cells will be

$$\frac{n(n^2-1)}{3} - n(n-1) = \frac{n(n-1)(n-2)}{3}$$ [9]

Distances d,c,r, and the Spread of Change in a Structure.

One way of studying the communication pattern of a structure is to assume a change of state as occurring in some cell and spreading by contact *throughout the structure.* Let us assume that at t^0 a change of state occurs in cell X_0, and that at t^1 the change has spread[10] to *all cells* X_1 *which touch* cell X_0, and that at t^2 the change has spread to all cells X_2 each of which touches at least *one cell of the* class of cells designated as X_1. If we assume further that the time intervals t^1-t^0, t^2-t^1, t^3-t^2, etc., are constant, we may describe some aspects of the communication pattern of a structure in terms of the time it takes a change to spread from place to place.

If we consider a simple chain in which cell A touches B, B touches C, C touches D, etc., and *assume a change* of state as originating in cell A, it is clear that the change will spread at the rate of one *cell per time unit* "t". If the change of state is assumed to be in cell A at time t^0, then at time t^1 the changed *state will be in both cells A* and B; in time t^2 it will be in the three cells A, B, and C; and at time t^n the change *will have spread to the* n^{th} *cell.* If, instead of a simple chain, we consider a structure such as that shown in Figure 8a with a change originating in cell A, the pattern of spread will be different.

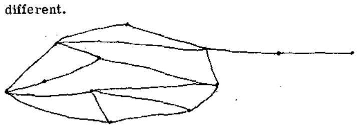


Figure 8a

Although the pattern of spread in a particular structure will be specific to that structure, it is possible to make certain *general statements with* respect to spread through time on the basis of some of the definitions and derivations made above:

1) A change of state not originating within a structure will start its spread within the structure from one or more of the cells comprising the outer region. Since the largest of the maximum distances between the outermost and innermost regions is equal to r (Definition 12), the t^n required for a change originating outside the structure to spread to the innermost region of the structure will be equal to t^r. The value r, however, was shown to vary with respect to d: d = r = d/2. Therefore, the t values necessary for a change to spread from the outermost to the innermost region under the conditions stated above will *range from* t^d *to* $t^{d/2}$.

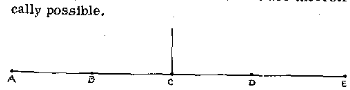
2) A change originating in the innermost region *of a structure, by the same reasoning* as in 1) above, will require from t^d to $t^{d/2}$ to spread to the outermost region of the structure.

3) A change originating in the most central region, will by definition require t^c to cover the entire structure. As in the case of t^r, the limits of t^c are also t^d to $t^{d/2}$. Since, however, c is defined in terms of the smallest p (Definition 14), t^c will be equal to or less than t^r.

In summation, one may say that whatever the structure a) a change may not spread throughout a structure in less than $t^{d/2}$, b) the smallest possible t for complete spread throughout a structure will be *obtained if the spread starts in the central region.*

Possible Applications of Some of the Concepts Above to Social Groups.

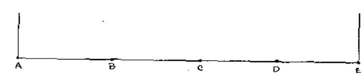
A study recently completed of a national minority organization indicated that the sub-group to which the organization as a whole was most responsive was also the sub-group in closest touch with the non-minority and anti-that-minority environment. Is this an example of a region of a social group which is at the same time the most central and the most outer region? What difference would it make to the life of this organization if its most central sub-group were also its most inner region? The pictures that follow show some of the variations that are theoretically possible.



A + E = innermost and peripheral region
$\overline{AE}$ = d   C = outer and central region

Figure 9

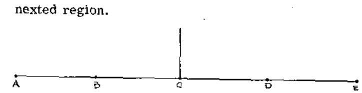
In Figure 9 we see a situation similar in some respects to the one mentioned above. The central region is also the outer region, and the innermost region is also the peripheral region. The innermost region is in two parts as widely separated as is possible.



C = innermost and central region
A + E = outermost and peripheral region
$\overline{AE}$ = d

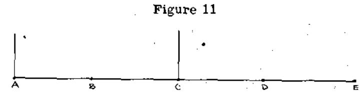
Figure 10

In Figure 10 we see a structure which appears in a *sense to be "opposite" that shown in Figure* 9. The central region is now furthest from the outside. The region *in contact with the "outside" is now an* unconnexted region.



C + D = central region and is partly outermost
A + F = *peripheral region*
F = most inner region

Figure 11



A + E = peripheral region and partly outer
A + C = outermost region
C = central region
E = innermost region

Figure 12

An interesting industrial situation about which the writer was told serves to illustrate a possible meaning of the kind of analysis of structures shown in Figures 9, 10, 11, 12. A group of women employed by a garment factory in a large city and working alongside each other constituted an informal work group. In addition to the fact that they were paid on the basis of their combined production, another circumstance tended to make this group a close social structure: all spoke Italian and only one of the women could speak English at all. Relations between this group and the management of the company regarding hours, wages, *working conditions took place* through the single English speaking member (this in

8. Let $\phi = n(n-1) + (n-1)(n-2) + \ldots\ldots\ldots + n-(n-1)\ (n-n)$

Then $\phi = \sum_{J=1}^{J=n} J(J-1) = \sum_{J=1}^{J=n} J^2 - \sum_{J=1}^{J=n} J$

$\sum_{J=1}^{J=n} J^2 = 1^2 + 2^2 + 3^2 + \ldots\ldots\ldots + n^2 = \frac{n(n+1)\ (2n+1)}{6}$ $\sum_{J=1}^{J=n} J = 1 + 2 + 3 + \ldots\ldots\ldots + n = \frac{n(n+1)}{2}$

$\phi = \frac{n(n+1)\ (2n+1)}{6} - \frac{n(n+1)}{2}$ $\phi = \frac{4n^3-4n}{12} = \frac{n}{3}(n^2-1) = \frac{n(n^2-1)}{3}$

9. Notice that the range of Σse_{xy} is $\frac{n}{3}$ times the range of si_x.

10. We will assume in this discussion that the change of state may spread indefinitely without diminution or increase.

spite of the fact that the plant was unionized). Clearly, this woman, with respect to communication with management, was the "outermost" member of the group. The inner structure of the group is not known, but one may speculate as to the effect upon communication within the group of the position she occupied within it.

If, as well as being the outermost member, she were also the most central, the group might be *represented as in Figure 9. It is difficult to imagine* that the English speaking member would be other than central with respect to communication which had of necessity to pass through her, although under certain conditions such a pattern might well exist. *In Figure 12a the English speaking member (E)* is shown in a noncentral position.

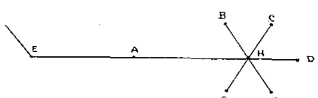


Figure 12a

It is interesting in passing to point out the importance of the position of the English speaking member with respect to the group's *perception of the* "outside." Even though the outermost region of a structure may be barred from policy formulation, it may still exercise a great effect upon policy decisions. To the extent that policy decisions are based upon information, as to the state of affairs "outside," withholding information, coloring or distorting it in transmission, or in other ways misrepresenting the state of the outside will fundamentally affect these decisions. In a sense, the "perceptual adequacy," loyalty, and morale of the outermost region of a group structures is crucial in maintaining optimum relations with the "outside."

While it is true that the structures under discussion are defined under assumptions which make their strict coordination with industrial or other hierarchical organizations impossible, they do provide a basis for the comparison of structural properties per se of certain types of organizations. There are some questions that can be answered quite specifically. *For instance, in an organization of a given type*, what is the maximum distance that there will be between any two individuals in it? Will this distance depend upon the number of people in the organization? the number of subordinates under each superior? the number of levels in the organization? the communication possibilities between subordinates? One might ask a different type of question: in a given organization where will the region of greatest centrality lie? Who will be in it? who will be most peripheral?

Some of the answers to these questions can be given. One may begin by specifying three types of structures for comparison.

1) an organization in which a subordinate *communicates only with his superior and with his* subordinates (Figure 13)

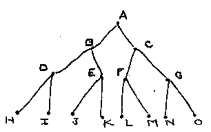


Figure 13

2) an organization in which a subordinate communicates with his superior, the other subordinates of his superior, and his own subordinates (Figure 14)

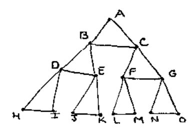


Figure 14

3) an organization in which a subordinate communicates with his superior, all subordinates on his own level, and with his own subordinates (Figure 15)

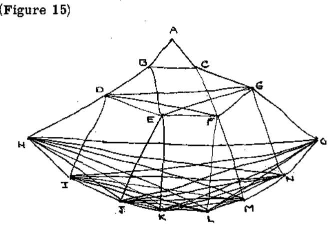


Figure 15

In these structures, as they are defined, the significant variable with respect to the values d and c is the number of levels rather than the number of cells. In Figure 13, for example, it is apparent on inspection that it doesn't matter how many subordinates each superior has connected to him; the d and c values will remain constant as long as the number of levels remains constant. The same thing is true of the structures shown in Figure 14 and Figure 15.

It would seem desirable, therefore, to express the d and c values in terms of the number of levels (L). The numerical values for d and c are shown in the tables that follow (Figure 16 and 16a)

Structure type	No. of cells	No. of levels	d	c
1 (Figure 13)	15	4	6	3
2 (Figure 14)	15	4	5	3
3 (Figure 15)	15	4	3	2

Figure 16

Structure Type	d in terms of L
1 (Figure 13)	2(L-1)
2 (Figure 14)	2(L-3/2)
3 (Figure 15)	L-1

Figure 16a

Looking at the structure shown in Figure 13, it is plain that the longest chain is that containing the cells H D B A C G O. it is also evident that the longest chain in any structure of this type will be the one going from the cells occupying the positions that cells H and O, or I and N, or J and M, etc., occupy *relative to the entire structure. The length* of the longest chain will always equal 2(L-1).

In the structure shown in Figure 14, the situation is the same except that in the chain H D B A C G O cell A may be omitted, due to the connection between cells B and C. Therefore, in structures of this type the longest chain will be equal to 2L-3.

In the structure shown in Figure 15, by the same process of reasoning, the longest chain will be L-1.

It appears, therefore, that the difference in d between the first two structures becomes insignificant as the number of levels increases, but that the difference between the d of the third type and the *other two becomes increasingly* great. For example, in a structure of ten levels the structures would show these d distances (Figure 17).

Structure Type	No. of levels	d
1 (Figure 13)	10	18
2 (Figure 14)	10	17
3 (Figure 15)	10	9

Figure 17

If we compare these structures with respect to the distance c, it is helpful again to express c in terms of the number of levels (Figure 18).

Structure Type	c in Terms of L
1 (Figure 13)	L-1
2 (Figure 14)	L-1
3 (Figure 15)	L/2[11]

Figure 18

In a structure of ten levels the c values would be as follows (Figure 19).

Structure Type	No. of Levels	c
1 (Figure 13)	10	9
2 (Figure 14)	10	9
3 (Figure 15)	10	5

Figure 19

Another way to compare these structures is that of the location of the central region. In the structure of the first type (Figure 13), the central region consists of the cell A. In the structure of the second type (Figure 14), the central region consists of the top two levels - cells A, B, and C. In a structure of the third type (Figure 15) the central region is the second level up from the bottom - cells D, E, F, G. As the number of levels increases, the central regions of *the first two types of structures* will remain located at the first and the first and second levels. The central region of the third type of structure behaves somewhat differently. If the number of levels is an even number, the central region is the L/2+1 level from the top. If the number of levels in the structure is an odd number, the central region consists of two adjacent levels - the one L+1/2+1 from the top.

Therefore, in a structure of ten levels the central regions would be located in the following levels (Figure 20).

11. In reading the value of L/2 read fractions to the next integer.

Structure Type	No. of Levels	Location of Central Region in Levels from the Top
1 (Figure 13)	10	1st level
2 (Figure 14)	10	1st + 2nd levels
3 (Figure 15)	10	6th level

Figure 20

Another comparison of these structures might be made in terms of the total pattern characteristics si_x and $\sum se_{xy}$. It was shown above that for a structure of n cells both of these values have definite limits. The structures under discussion consist of 15 cells. The ranges of si_x and $\sum se_{xy}$ (see pages 23 and 25) for n=15 are shown below (Figure 21).

When n = 15

Lower Limit		Upper Limit
28	si_x	210
210	$\sum se_{xy}$	1120

Figure 21

The relative positions of the three structures in these ranges are shown below (Figure 22).

Structure Type	si_x	se_{xy}
1 (Figure 13)	28	736
2 (Figure 14)	42	594
3 (Figure 15)	98	374

Figure 22

In interpreting these figures it may be helpful to think in averages. In such terms $si_x/15$ would be the average number of neighbors per cell, and $se_{xy}/15$ would be the average distance between cells (Figure 23).

Structure Type	Average No. of neighbors	Average Distance between cells
1 (Figure 13)	1.9	49.1
2 (Figure 14)	2.8	39.6
3 (Figure 15)	6.5	24.9

Figure 23

Up to this point the distance r has not entered the discussion because the structures under consideration have consisted of closed cells only. If a structure has no open cells, r has no value. What would happen if an open cell were added? For instance, assuming that some person is to be given the function of introducing from the "outside" changes which should spread as quickly as possible throughout the organization, in the structure shown in Figure 13 it is evident that he should be attached to the organizational head - cell A, the most central region. In the structure shown in Figure 15 such an attachment would be one of the poorest to be made, since in that structure the organizational head is in the peripheral rather than in the central region.

The three types of structures discussed above are simple models which one would hardly expect to find duplicated in an existing social organization. Nevertheless, the kind of analysis that has been attempted is useful. It suggests, for instance, experimental group structures for the study of communication, and affords concepts for their structural evaluation and comparison. Recently, several trial runs of such an experimental setting were made by the writer. A group was given a task to perform which necessitated a relatively high order of communication between individuals. The pattern of communication was experimentally controlled by limiting it to a previously structured telephone system. The results of the few trials that were made suggested rather strongly that a) perception of a co-worker's ability and personality, b) degree of confidence in the successful accomplishment of the task, and c) the pace at which work could be done comfortably were all greatly affected by the structural properties of the communications pattern under which the group operated.

Another value of the kind of analysis attempted above is that it raises clearly the problems of coordination. For instance, in a typical industrial organization what may be defined as the outermost region? Would the sales force be in the outermost region? The advertising department? The employment office? The receiving room? The receptionist? Obviously, answering these questions will depend partly on what kind of communicative material is being considered. The communication pattern in a group structure will be different for a funny story than for news of a new stock issue. In the same way, the nature of "outside" will be different if one thinks of communication from the outside to the structure of baseball news as against stock market trends.

It is helpful, also, to consider the definition of "outer region". The outermost region of a structure is composed of all the cells which are "open" relative to the structure (Definition 11). An open cell of a structure is any cell whose "boundary" is not contained in the structure (Definition 3). The boundary of a cell consists of all other cells touching that cell (Definition 1). In the case of the re-

ceiving-shipping department of a company, if materials were received from and shipped to branches of the parent company only, one might conclude from the definitions given above that the department was not part of the outer region. The psychological meaning of "shipping and receiving" would probably change if this department were to become an outer region. This difference might express itself in terms of department discipline, standards of performance, work organization, and attitude toward the "customer".

DIRECTION WITHIN STRUCTURES

Just as in the case of distance, psychological direction is impossible to define in terms of Euclidian concepts except in very special circumstances; when psychological and physical movement toward a goal coincide. In many instances where the psychological goal is the reaching of a physical object, the direction of observable bodily movement may be quite different from the psychological direction to the goal. A person traversing a familiar maze may be moving directly away, physically, from the goal, although moving psychologically closer. Other types of changes of position which have a clear psychological direction - such as becoming a member of a social group - may have no physical correlate whatsoever.

In his hodological geometry, Lewin approached the problem of direction from the same basis as that of distance.

> "The distance a,b refers to the length of (the) distinguished path from a to b ...... The concept of direction in hodological space follows the same pattern. It also refers to a path between two regions A and B. Generally there are many such paths possible between two such regions and it is necessary to select one path which will determine the direction (of the path A to B)."[12]

The problem of selecting the distinguished (shortest) path has been discussed above. As in the treatment of distance, this thesis attempts the development of a concept of direction on the basis of a "minimal regions" definition of shortest path. The following ideas are developed below:

a) Steps toward or away, or neither toward or away from specified cell A_o.

b) A straight line path toward A_o.

c) A straight line path from one cell A_o to another cell B_o.

d) Direction of movement along a straight line path from A_o to B_o.

e) A general statement of the definition of "straight line path".

Steps Toward or Away from a Specified Cell

A cell A is selected as a point of reference, and all other cells in the structure are referred to in terms of their distance from A. This relationship may be conveniently indicated by the use of subscripts. Let the cell which is selected as the point of reference be designated as A_o and all cells which are a distance of 1 away from A_o be designated as A_1, etc. Thus a structure is stratified into "layers" in terms of distance from A_o. A step from any cell in the structure to an adjoining cell may be said to be toward A_o if the step is from a higher to a lower subscript. Thus, the step A_n, A_{n-1} is a step toward A_o. The step A_n, A_{n+1} is a step away from A_o. The step A_n, A_n is a step neither toward nor away from A_o (see Figure 24).

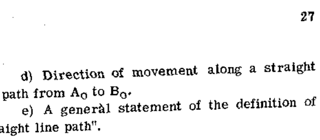


Figure 24

Step A_1 A_2 is away from A_o

Step A_2 A_1 is toward A_o

Step A_2 A_2 is neither toward nor away from A_o

A Straight Line Path Toward A_o.

Any chain of cells so arranged that successive steps along it show a decreasing subscript is considered to be a straight line path toward A_o. All such chains may be expressed in the form

$$A_n, A_{n-1}, A_{n-2}, \ldots\ldots, A_2, A_1, A_o$$

In a given structure there will be at least one straight line path from every cell not A_o to A_o (see Figure 25). This follows simply from the method used to assign subscripts. If a cell not A_o bears the subscript A_x, it must by definition be touching at least one cell with a subscript of A_{x-1}. The cell A_{x-1} must, in turn, be touching a cell with the subscript A_{x-2}, etc.

12. Op. cit., p. 6

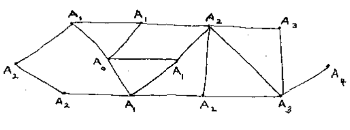


Figure 25

From A_4 there are three different straight line paths to A_o.

A Straight Line Path from A_o to B_o.

If a structure is stratified with respect to both a cell A and a cell B, each cell in the structure will have an "A" subscript and a "B" subscript written in the form A_xB_y. In a simple chain structure of five cells the subscript would be shown in Figure 26.

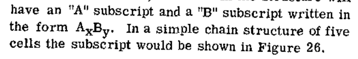


Figure 26

A straight line path from cell A_o to cell B_o is defined as a chain that may be written A_oB_n, A_1B_{n-1}, A_2B_{n-2}, ..., $A_{n-2}B_2$, $A_{n-1}B_1$, A_nB_o.

Direction of Movement Along A Straight Line Path from A to B.

The direction of movement along a straight line path from A to B may be indicated by assuming all A subscripts to be positive and all B subscripts to be negative. An algebraic summation of subscripts for each cell may then be made. A summation of each step of the sequence A_oB_n, A_1B_{n-1}, A_2B_{n-2} ....., $A_{n-2}B_2$, $A_{n-1}B_1$, A_nB_o yields the following result: -n, -n+2, -n+4, ..., n-4, n-2, n. In terms of the above sequence, it may be said that in a straight line path from A to B the absolute difference between algebraically summated subscripts is equal to 2. A step along this path toward B gives a subscript differential of +2 or (-n+4)-(n+2); a step along this path toward A gives a subscript differential of -2 or (-n+2)-(-n+4).

It was stated above that there is at least one straight line path from any cell not A_o to A_o. Since the cell B_o is one cell of the class of cells not A_o, it follows that there is at least one straight line path from B_o to A_o. By similar reasoning there is at least one straight line path from A_o to B_o. It may be shown, also, that there is a path which is both a straight line path from A_o to B_o, and a straight line path from B_o to A_o. The demonstration is as follows: take any straight line path from A_o to B_o. The cell B_o will have an A subscript of some value x. By definition, there must be some cell touching this cell A_xB_o which has the subscript of A_{x-1}. The B subscript of the cell A_{x-1} will necessarily be B_1. There must be some cell touching the cell $A_{x-1}B_1$ which has the A subscript of A_{x-2}. The B subscript of such a cell cannot be B_o since B_o is a unique cell in the structure. It cannot be B_1 because:

a) Any cell B_1 touches cell B_o.
b) Cell B_o has an A subscript of x.
c) But the subscript of adjoining cells may vary only by 1.
d) Therefore, the cell A_xB_o may not touch the cell $A_{x-2}B_1$ since the A subscripts differ by more than one.
e) Therefore, the cell A_{x-2} must bear the subscript B_2.[13]

In the same way, it may be shown that in a chain from B_o to A_o in which the A subscripts decrease at each step (a straight line path from B_o to A_o), the B subscripts will increase at each step from B_o to A_o (a straight line path from A_o to B_o).

Constant Differential Paths Other than Path A_o B_o.

If a straight line path between A_o and B_o is defined in terms of a constant subscript differential at 2, what about paths with constant differentials at some other value, if any such paths exist?[14] The problem may be stated in this way: in a structure in which no restrictions are placed upon the number of cells or their pattern of connection, how many, if any, paths with different constant subscript differentials may there be? The following analysis appears to answer this question.

In a structure that has been stratified with respect to some cell A and some cell B let the cell A_xB_y (contained in the chain A_oB_o) be selected. Since the A and B subscripts in any adjoining cell may vary only by one (or remain constant), the subscripts for all cells touching A_xB_y may be written (Figure 27).

13. It is possible, of course, for the cell $A_{x-2}B_1$ to exist in the structure, but it may not touch the cell A_xB_o.

14. Except where stated otherwise, in the discussion that follows it is assumed that the structures dealt with are stratified with respect to only two points of reference - A_o and B_o.

A_xB_y	A_xB_{y+1}	A_xB_{y-1}
$A_{x+1}B_y$	$A_{x+1}B_{y+1}$	$A_{x+1}B_{y-1}$
$A_{x-1}B_y$	$A_{x-1}B_{y+1}$	$A_{x-1}B_{y-1}$

Subscripts of all cells touching cell A_xB_y

Figure 27

Assuming that the A subscripts are positive and the B subscripts are negative, the summation of the subscripts is shown in Figure 28.

x-y	x-y-1	x-y+1
x-y+1	x-y	x-y+2
x-y-1	x-y-2	x-y

Summation of subscripts shown in Figure 27.

Figure 28

The difference between the subscripts of each of these cells and A_xB_y may now be calculated (Figure 29).

0	-1	+1
+1	0	+2
-1	-2	0

Subscript differential for every possible step from A_xB_y

Figure 29

It may be said, therefore, that in a structure stratified with respect to two cells and with no restriction placed upon the number of cells or the pattern of connection, paths may be distinguished with constant subscript differentials of o,-1,+1,-2,+2. It is helpful at this point to draw a structure showing the subscript differentials derived above (Figure 30a, 30b).

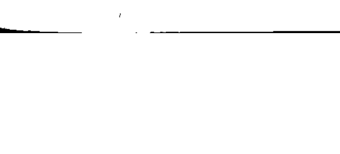


Figure 30a

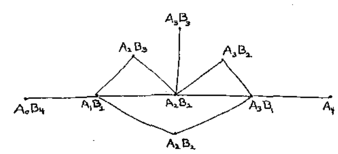


Figure 30b*

*The structures shown in figures 30a and 30b are identical. Two figures are used for convenience in indicating path differentials. For instance, the step C,D is translatable into A_1B_3, A_2B_3 which is equal to summated subscripts -2, -1. Transposing these quantities in order to get the subscript differential of the step: -1(-2) = -1+2 = +1.

The table below (Figure 31) gives some of the steps and their subscript differentials.

Step	Subscript Differential
G E	0
G H	0
G D	-1
G F	+1
G C	-2
G I	+2

Figure 31

A General Definition of Straight Line Path.

Any chain is considered to be a straight line path if it has a constant subscript differential at each step. In a given structure the subscript differential of a given step may change if the location of A_o and/or B_o is changed. It should be understood, therefore, that a chain is a straight line path relative to the A_oB_o matrix that is being used.

CONCLUSION

The mathematical model presented here is admittedly in an early state of development, and the problem of coordinating the mathematical concepts to psychological data is still to be met. The main objective of this paper is to define a possible geometry for dealing with psychological space, and to explore in a limited way the consequences of a particular set of assumptions and definitions.

The fruitfulness of a model such as the one presented above may be judged not only by the questions it answers, but also by the questions it prompts one to ask. In a logical system as undeveloped as this one, the choice of "next steps" is sure to be greatly colored by personal interest however "logical" the choices are claimed to be. But the setting down of the questions that would, to the writer, be next steps is one way of indicating possible continuations.

The questions divide themselves naturally into two main groups: questions with respect to the mathematical development of the model, and questions with respect to the problem of coordination.

1) Mathematical development problems.

a) Although the distances c and r have the same limits in terms of d, it has been shown that they do not vary within those limits in the same way. Why? What is the precise relationship between c, r, and d?

b) A structure of n cells and with a constant si_x may take more than one form. Some of these froms have different $\sum se_{xy}$ values. Likewise, a structure of n cells and with a constant $\sum se_{xy}$ may take more than one form, and some of these forms have different si_x values. What precisely is the relationship between si_x and $\sum se_{xy}$?

c) There appear to be many evidences that the length of the straight line path A_o to B_o in a structure limits the length of straight line paths other than paths with a zero differential. Is this true, and, if so, how does the limitation operate?

2) Coordination problems.

a) It has been shown that a path within a structure may have a direction. A psychological force is also assumed to have a directional value. Can the direction of a psychological force on a person at a cell B_x toward the cell A_y be coordinated to the direction of the straight line path from B_x to A_y?

b) To what psychological entity should a cell be coordinated? If cells are coordinated to activities, goals, social position, etc., to what should regions be coordinated?

c) In order to assign directional values to paths, certain cells (A_o and B_o) were designated as reference points. To what should these reference points be coordinated in a life space? It would appear that the coordination should be made to some part in the life space sufficiently outstanding so that all other parts are seen in relation to it. Goal regions seem an obvious choice, but many other useful coordinations seem possible.

Obviously, these are only a few of the questions that could be set down, and it is impossible to predict what course further work will actually take. For the psychologist there is an understandable urge to proceed with the business of coordination. It may be, however, that the possibility and fruitfulness of coordination are very different at different stages of development of a model, and that further mathematical development of the model is the quickest way to a useful coordination with psychological fact.